

Symmetric Key Encryption for Arbitrary Block Sizes from Affine Spaces

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Abstract: A symmetric key encryption scheme is described for blocks of general size N that is a product of powers of many prime numbers. This is accomplished by realising each number (representing a message unit) as a point in a product of affine spaces over various finite fields. Then algebraic-geometric transformations on those spaces is transported back to provide encryption. For a specific block size $< 2^{128}$ we get more than 2^{5478} keys.

Keywords: block ciphers; symmetric key encryptions; FPE; affine spaces; Jonquières automorphisms

Introduction

A typical block cipher takes the set $\mathcal{M} = \{0, 1, 2, \dots, N - 1\}$, the set of message units, and provides one permutation of that set corresponding to each element (*a secret key*) of another set called the key space. The larger the key space, the more difficult it is to break the cryptosystem by exhaustive search for keys.

For ease of implementation in electronic hardware, usually the cardinality N is taken as a power of 2. When $N = 2^n$, these are called n -bit block ciphers.

The well-known industry standard cipher, AES encryption, provides three variants that are 128-bit, 192-bit and 256-bit block ciphers. (That is, they permute sets of huge sizes such as 2^{128}).

There are situations where one needs permutations of sets of size different from powers of 2, and that branch of cryptology is called *format-preserving encryption* (FPE). (See Black and Rogaway [BR02], Brightwell and Smith [BS97] and [FIPS74]). FPE, for example, tries to encrypt a 16-digit credit card number into something that again looks like a credit card number.

In this paper we describe a construction of block cipher for sets of size N with N factorizable into many prime powers.

The novelty in our scheme is the choice of geometric model for the set of message units. It is a product of *affine spaces*. Being an uncomplicated affine variety the task of mounting points of the message space onto this geometric object is a simple one. The transformations on our geometric model arise from Jonquières automorphisms: they are our preferred choice due to their ready invertibility though there are different kinds of automorphisms there.

We simultaneously use Jonquières automorphisms over all finite fields \mathbf{F}_p corresponding to every prime divisor p of the block size N , and then patch them to yield a permutation on the block by appealing to Chinese Remainder Theorem.

This encryption scheme makes available an immense collection of permutations (i.e., a huge keyspace). To compare, while AES provides 2^{128} keys, this method for a slightly smaller block size (but chosen conveniently), even with an artificial restriction to smaller keys (“*polynomials of degree ≤ 5* ”), yields a keyspace of size bigger than 2^{5478} . Moreover, these permutation calculations can be parallelized for speed.

Here are the salient features of this encryption scheme:

- As we map the message units with the points of affine spaces, one layer of complexity at pre-encryption stage is eliminated providing considerable simplification.
- Affine spaces as opposed to elliptic curves and abelian varieties are not only conceptually simpler objects of algebraic geometry but also come with a vast collection of automorphisms compared to them, thereby affording a large keyspace meeting an important requirement of information security.
- Block ciphers for blocks of size different from powers of a single prime are provided.
- All the bijections and permutations used here are explicit, natural mathematical constructions, and they are easily implementable as computer programs. (We have implemented this algorithm for numbers of the form $N = p^4 q^4$ as a Python program with 150 lines of code).
- Computations are parallelizable: for any permutation provided by this scheme, where any two different elements should be sent can be computed without the knowledge of each other.

Now we hasten to add that though theoretical description is available for blocks whose size N has two or more distinct prime divisors, from the viewpoint of utility value, only a certain restricted numbers N might be suitable. We provide such a suitable number to compare the keyspace size with 128-bit encryption.

For convenience and for avoiding notational clutter we describe our scheme in the simpler case where block size N is of the form $N = p^r q^s$, with p, q distinct primes. Working out the general case of N having three or more prime factors is straightforward.

This paper is organized as follows:

- In Section 1 we recall the definition of Jonquières automorphisms which is a fundamental ingredient in our scheme. This focusses on sets whose cardinality is a prime power.
- In the second section we provide the full description of our encryption scheme which builds on top of Section 1 and uses Chinese Remainder Theorem for weaving together the automorphisms coming from various prime-power divisors of the block size.
- In Section 3 an alternative description is provided in an algorithmic fashion to aid in computation.
- Finally in Section 4 we illustrate with an example that uses blocks of a specific size slightly less than 2^{128} and compute the size of the key space.

1 Jonquières Automorphisms

Consider the affine space \mathbf{A}^n of dimension n , the set of n -tuples of elements over some field \mathbf{F} . We avoid calling it a vector space for the simple reason that we will be doing non-linear operations on this set.

We use (x_1, x_2, \dots, x_n) as co-ordinates of a point in this space. As an affine algebraic variety \mathbf{A}^n admits many automorphisms, besides the translations and linear automorphisms. Jonquières has defined a ‘triangular’ family of polynomial automorphisms ([J1864]). To define a Jonquières automorphism one has to first choose $n - 1$ polynomials P_1, P_2, \dots, P_{n-1} with coefficients in \mathbf{F} such that P_1 involves just one variable x_1 , P_2 involves just x_1 and x_2 . In general the polynomial P_i is taken to be involving only the first i variables x_1, x_2, \dots, x_i . Along with this we also need non-zero scalars from the base field, $a_i \in \mathbf{F}^*$, for $1 \leq i \leq n$.

Lemma 1 (E. de Jonquière, 1864) Assume a_i and the polynomials P_i are as above. Further assume that for any of these P_i the degree is less than p . Then the map $J: \mathbf{A}^n \rightarrow \mathbf{A}^n$ defined by sending (x_1, x_2, \dots, x_n) to (y_1, y_2, \dots, y_n) by the formula

$$\begin{aligned} y_1 &= a_1 x_1 \\ y_2 &= a_2 x_2 + P_1(x_1) \\ y_3 &= a_3 x_3 + P_2(x_1, x_2) \\ &\vdots \\ y_n &= a_n x_n + P_{n-1}(x_1, x_2, \dots, x_{n-1}) \end{aligned}$$

is an automorphism of the affine space as an affine algebraic variety.

Proof: The condition on the degree is to avoid terms of the form x_i^p which bring inseparability issues. In all real world cryptographic applications we need to choose $p > 100$, and degrees < 10 .

To show that this is an automorphism one should exhibit an inverse function and show that the inverse is also a polynomial function. Inverting is not difficult, one simply imitates the procedure for solving a triangular system of *linear* equations.

The inverse J^{-1} can be computed *sequentially* by the formulas below:

$$J^{-1}(y_1, y_2, \dots, y_n) = (x_1, x_2, \dots, x_n) \quad \text{with}$$

$$\begin{aligned} x_1 &= a_1^{-1} y_1 \\ x_2 &= a_2^{-1} (y_2 - P_1(x_1)) \\ x_3 &= a_3^{-1} (y_3 - P_2(x_1, x_2)) \\ &\vdots \\ x_n &= a_n^{-1} (y_n - P_{n-1}(x_1, x_2, \dots, x_{n-1})) \end{aligned}$$

This proves the lemma and much more: the inverse of J is also a Jonquière automorphism (see the first remark below).

Remark 1 The data consisting of scalars a_i 's and polynomials P_i 's essentially form the encryption key for our scheme. To justify the name *symmetric key* encryption we can see that in the opposite direction the scalars b_i 's are simply the inverses of a_i 's modulo p , and the polynomials in the opposite directions are got by changing the signs of all the coefficients of P_i 's followed by multiplication by b_i 's. The upshot of this is that the decryption key is readily obtained from the encryption key and they are mutual inverses as required.

Remark 2 Other than the fact that these P_i 's should involve only the variables x_1 to x_i (*backward-mixing*) there is no restriction on them in order to define an automorphism. So one has a huge collection of Jonquière's automorphisms that can be readily written down. (see Lemma 3 below).

Remark 3 For practical considerations to enable good mixing one should also introduce *forward-mixing*. For this purpose we take two such Jonquière's automorphisms, apply one of them first, follow it up with the reversal map, $(x_1, x_2, \dots, x_{n-1}, x_n) \mapsto (x_n, x_{n-1}, \dots, x_2, x_1)$ and then apply the second Jonquière's automorphism. (Alternatively this can be understood as analogous to carrying out an upper triangular transformation followed by a lower triangular transformation).

Remark 4 Any function $\mathbf{F}_p \rightarrow \mathbf{F}_p$ is a polynomial function by Lagrange interpolation formula. Higher dimensional analogue of this is also true. Any function from \mathbf{A}_p^n to itself is a polynomial, actually a polynomial of degree at most $(p-1)^n$. I thank `user9072` of the internet forum www.mathoverflow.com for pointing out the validity of this generalization.

Lemma 2 Let $\phi_1 = (F_1, F_2, \dots, F_n): \mathbf{A}_p^n \rightarrow \mathbf{A}_p^n$ and $\phi_2 = (G_1, G_2, \dots, G_n): \mathbf{A}_p^n \rightarrow \mathbf{A}_p^n$ be two (polynomial) automorphisms. If total degree $< p$ for every F_i and G_j and if ϕ_1, ϕ_2 are different as polynomials (i.e., at least some pair of corresponding coefficients are different) then $\phi_1 \neq \phi_2$ as *functions* on \mathbf{A}_p^n .

Proof: Suppose ϕ_1 and ϕ_2 are one and the same as functions. Then, in particular, $F_1(x_1, x_2, \dots, x_n) \equiv G_1(x_1, x_2, \dots, x_n)$. Specializing all the x_j , $j \geq 2$ at 1, we get the following equality of univariate polynomial functions over \mathbf{F}_p : $F_1(x_1, 1, 1, \dots, 1) = G_1(x_1, 1, 1, \dots, 1)$. This shows that the difference between these polynomials in x_1 is of degree at least p , as it has all the elements of \mathbf{F}_p as its roots. QED

Lemma 3 The number of ‘upper triangular’ Jonquière's automorphisms over a given prime field using low degree polynomials are as in the table below:

Affine Space	Degree	Number of Automorphisms
\mathbf{A}_p^4	≤ 3	$p^{34}(p-1)^4$
\mathbf{A}_p^4	≤ 4	$p^{55}(p-1)^4$
\mathbf{A}_p^4	≤ 5	$p^{83}(p-1)^5$
\mathbf{A}_p^5	≤ 3	$p^{69}(p-1)^5$
\mathbf{A}_p^5	≤ 4	$p^{125}(p-1)^5$
\mathbf{A}_p^5	≤ 5	$p^{209}(p-1)^5$

Proof: This follows from the well-known formula for the number of monomials of degree d in n variables.

2 Construction of the Block Cipher

As stated in the introduction we take for simplicity $N = p^r q^s$. Our object is to produce explicitly a large family of constructible permutations of the set of message units, $\mathcal{M} = \{0, 1, 2, \dots, N - 1\}$.

Our idea can be summarised as below:

Step (i) to identify \mathcal{M} with a geometric object, viz. a cartesian product of affine spaces (over different prime fields) through explicit bijections.

Step (ii) apply Jonquière's automorphisms independently in each of the affine spaces forming the terms of the cartesian product above.

Step (iii) transport the product of Jonquière's automorphisms back to the message space \mathcal{M} through the inverse of the bijections mentioned in Step (i). It is simply retracing the bijections of Step (i).

2.1 Details of Step (i): Message Space to Affine Space

For this we regard \mathcal{M} as the commutative ring of integers modulo N . Our bijection of Step (i) is obtained as a composition of two bijections: first one denoted by ψ , is a ring isomorphism, and the second one denoted by δ is a set-theoretic bijection as indicated below:

$$\mathcal{M} = \mathbf{Z}/N\mathbf{Z} \xrightarrow{\psi} \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z} \xrightarrow{\delta} \mathbf{A}_p^r \times \mathbf{A}_q^s$$

Description of ψ, ψ^{-1} : ψ is simply the calculation of the two remainders of a number for division by p^r and q^s .

$$\psi(m) = (m \bmod p^r, m \bmod q^s)$$

The inverse, ψ^{-1} , is the map provided by Chinese Remainder Theorem.

Description of δ : First we define a function δ_p for $a < p^r$ as a vector formed by its digits in base p expansion:

$$\delta_p(a) = (\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \quad \text{for } a = \sum_{j=0}^{r-1} \alpha_j p^j, \quad 0 \leq \alpha_j < p.$$

Similarly $\delta_q(b)$ is defined using base q digits. Now δ is defined as

$$\begin{aligned}\delta: \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z} &\longrightarrow \mathbf{A}_p^r \times \mathbf{A}_q^s, \\ \delta(a, b) &= (\delta_p(a), \delta_q(b))\end{aligned}$$

Inverse δ_p^{-1} is even easier to compute. Interpreting a vector with all components integers less than p as the base- p digits, this will represent a number less than p^r . (Similarly for δ_q^{-1}).

Example

Take $N = 5000 = 2^3 5^4$. Let us first calculate $\psi(471)$. As $471 \pmod{2^3} = 7$ and $471 \pmod{5^4} = 96$ we have $\psi(471) = (7, 96)$.

Now $\delta((7, 96)) = ((1, 1, 1), (0, 3, 4, 1))$ (because 7 in binary is 111 and 96 in base 5 is 341. As $5^4 | N$, we have to write residues modulo 5^4 as 4-digit numbers in base 5, inserting leading zeros where needed).

2.2 Details of Step (ii): Geometric Transformation within the Affine Spaces

We take two pairs of Jonquières automorphisms (J_p^1, J_q^1) and (J_p^2, J_q^2) which makes up our encryption key in this case. Each pair gives rise to a bijection $J = J_p \times J_q: \mathbf{A}_p^r \times \mathbf{A}_q^s \rightarrow \mathbf{A}_p^r \times \mathbf{A}_q^s$, i.e., a cartesian product of Jonquières automorphisms $J_p: \mathbf{A}_p^r \rightarrow \mathbf{A}_p^r$, and $J_q: \mathbf{A}_q^s \rightarrow \mathbf{A}_q^s$.

Key Selection To get a key one has to select finite sequences (of appropriate length) of random integers in the ranges $[0, p-1]$ and $[0, q-1]$ respectively to be used as coefficients of polynomials which make up the four Jonquières automorphisms, $J_p^1, J_p^2, J_q^1, J_q^2$.

Let us assume such a selection has been made giving rise to a key K consisting of two pairs of Jonquières automorphisms $K: (J_p^1, J_q^1), (J_p^2, J_q^2)$. Define reversal map as $\mathbf{A}_p^r \times \mathbf{A}_q^s \rightarrow \mathbf{A}_p^r \times \mathbf{A}_q^s$ by

$$(\alpha_0, \alpha_1, \dots, \alpha_{r-1}; \beta_0, \beta_1, \dots, \beta_{s-1}) \xrightarrow{\text{rev}} (\alpha_{r-1}, \alpha_{r-2}, \dots, \alpha_0; \beta_{s-1}, \beta_{s-2}, \dots, \beta_0)$$

We apply the first pair (J_p^1, J_q^1) to an element of the product $\mathbf{A}_p^r \times \mathbf{A}_q^s$. Then apply reversal and finally apply the second pair of Jonquières automorphisms (J_p^2, J_q^2) . This completes Step (ii).

2.3 Details of Step (iii): Back to Message Space from Affine Space

In this stage we travel backwards to message space.

$$\mathbf{A}_p^r \times \mathbf{A}_q^s \longrightarrow \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z} \longrightarrow \mathbf{Z}/N\mathbf{Z} = \mathcal{M}$$

Given an element in \mathbf{A}_p^r take the components as base p digits and compute the number $< p^r$ represented by it. Similarly do the same for the element from \mathbf{A}_q^s getting a number less than q^s .

The second leg of our return journey is δ^{-1} . By Chinese remainder theorem we get a unique number less than $N = p^r q^s$ using the two numbers just obtained in the first leg of the journey taking us back to \mathcal{M} .

2.4 The Full Picture

Now we get an encryption, i.e., a permutation E_K of $\mathcal{M} = \mathbf{Z}/N\mathbf{Z}$ corresponding to the choice of key K , through a sequence of compositions as described by the commutative diagram shown in Figure 1.

$$\begin{array}{ccc}
 \mathbf{Z}/N\mathbf{Z} & \xrightarrow{\psi} & \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z} \\
 & & \downarrow \delta \\
 & & \mathbf{A}_p^r \times \mathbf{A}_q^s \\
 & & \downarrow J_p^1 \times J_q^1 \\
 & & \mathbf{A}_p^r \times \mathbf{A}_q^s \\
 & & \downarrow \text{rev} \\
 & & \mathbf{A}_p^r \times \mathbf{A}_q^s \\
 & & \downarrow J_p^2 \times J_q^2 \\
 & & \mathbf{A}_p^r \times \mathbf{A}_q^s \\
 & & \downarrow \delta^{-1} \\
 & & \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z} \\
 \mathbf{Z}/N\mathbf{Z} & \xleftarrow{\psi^{-1}} & \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/q^s\mathbf{Z}
 \end{array}$$

Figure 1: Schematic Description of the Encryption Algorithm

3 Algorithm

Set-up: $\mathcal{M} = \{0, 1, 2, \dots, N-1\}$, with $N = p^r q^s$

Encryption Key: Two ordered sets of polynomials $\{P_1, P_2, \dots, P_{r-1}\}$, and $\{Q_1, Q_2, \dots, Q_{s-1}\}$, with the first set having coefficients in \mathbf{F}_p , the second set in \mathbf{F}_q , with the i th polynomial in both sets involving only the first i variables; and scalars $a_i \in \mathbf{F}_p^*, b_j \in \mathbf{F}_q^*$ for $0 \leq i \leq r-1$, $0 \leq j \leq s-1$. Denote this half of key datum by K , i.e.,

$$K = (P_1, P_2, \dots, P_{r-1}, a_0, a_1, \dots, a_{r-1}, Q_1, Q_2, \dots, Q_{s-1}, b_0, b_1, \dots, b_{s-1}).$$

Similarly the other half of key datum is:

$$K' = (P'_1, P'_2, \dots, P'_{r-1}, a'_0, a'_1, \dots, a'_{r-1}, Q'_1, Q'_2, \dots, Q'_{s-1}, b'_0, b'_1, \dots, b'_{s-1}).$$

Keyspace A single key for our encryption scheme, in the general case where $N = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ comprises two pairs of k -tuples of Jonquières : $(J_{p_1}^1, J_{p_2}^1, \dots, J_{p_k}^1; J_{p_1}^2, J_{p_2}^2, \dots, J_{p_k}^2)$ and $(F_{p_1}^1, F_{p_2}^1, \dots, F_{p_k}^1; F_{p_1}^2, F_{p_2}^2, \dots, F_{p_k}^2)$.

Algorithm: Given a number $m \in \mathcal{M}$ its encrypted value $E_K(m)$ for given key K is computed by the steps below.

- Calculate the remainders m_p, m_q :

$$m_p \equiv m \pmod{p^r}; \quad m_q \equiv m \pmod{q^s}$$

- Extract the digits α_i of m_p and β_j of m_q in the bases p and q respectively.

$$\sum_{i=0}^{r-1} \alpha_i p^i = m_p; \quad \sum_{j=0}^{s-1} \beta_j q^j = m_q; \quad (\alpha_i < p, \beta_j < q \forall i, j)$$

- Transform the alphas and betas by respective Jonquières automorphisms.

$$\begin{aligned} (\alpha'_0, \alpha'_1, \dots, \alpha'_{r-1}) &= (a_0 \alpha_0, a_1 \alpha_1 + P_1(\alpha_0), a_2 \alpha_2 + P_2(\alpha_0, \alpha_1), \\ &\quad \dots, a_{r-1} \alpha_{r-1} + P_{r-1}(\alpha_0, \alpha_1, \dots, \alpha_{r-2})) \\ (\beta'_0, \beta'_1, \dots, \beta'_{s-1}) &= (b_0 \beta_0, b_1 \beta_1 + Q_1(\beta_0), b_2 \beta_2 + Q_2(\beta_0, \beta_1), \\ &\quad \dots, b_{s-1} \beta_{s-1} + Q_{s-1}(\beta_0, \beta_1, \dots, \beta_{s-2})) \end{aligned}$$

- Reverse the components and apply the other pair of Jonquières automorphisms.

$$\begin{aligned}
(\alpha''_0, \alpha''_1, \dots, \alpha''_{r-1}) &= (a'_0 \alpha'_{r-1}, a'_1 \alpha'_{r-2} + P'_1(\alpha'_{r-1}), a'_2 \alpha'_{r-3} + P'_2(\alpha'_{r-1}, \alpha'_{r-2}), \\
&\quad \dots, a'_{r-1} \alpha'_0 + P'_{r-1}(\alpha_{r-1}, \alpha_{r-2}, \dots, \alpha_1)) \\
(\beta''_0, \beta''_1, \dots, \beta''_{s-1}) &= (b'_0 \beta'_{s-1}, b'_1 \beta'_{s-2} + P'_1(\beta'_{s-1}), b'_2 \beta'_{r-3} + P'_2(\beta'_{s-1}, \beta'_{s-2}), \\
&\quad \dots, b'_{s-1} \beta'_0 + P'_{s-1}(\beta_{s-1}, \beta_{s-2}, \dots, \beta_1))
\end{aligned}$$

- Assemble the individual digits α''_i 's and β''_j 's into numbers $m'_p (< p^r)$, $m'_q (< q^s)$, (r and s digit numbers in base p and q respectively):

$$m'_p = \sum_{i=0}^{r-1} \alpha''_i p^i; \quad m'_q = \sum_{j=0}^{s-1} \beta''_j q^j$$

- Compute by Chinese Remainder Theorem the unique number $m' < N$ satisfying

$$m' \equiv m'_p \pmod{p^r}; \quad m' \equiv m'_q \pmod{q^s}$$

For this purpose one can, using Extended Euclidean Algorithm, pre-compute (once) and store e_p, e_q satisfying

$$e_p \equiv 1 \pmod{p^r}, e_p \equiv 0 \pmod{q^s}; \quad e_q \equiv 1 \pmod{q^s}, e_q \equiv 0 \pmod{p^r}$$

Then compute m' by

$$m' = (m'_p e_p + m'_q e_q) \pmod{N}$$

.

The m' obtained in the last step is the encrypted value $E_K(m)$.

4 Comparison with AES-128

Computations were done with SAGE Version 6.10. The block size of AES is:

$$2^{128} = 340282366920938463463374607431768211456$$

We searched for a number that had every prime dividing at least to the 5th power and *close to* 2^{128} . By trial and error we arrived at

$$p_1 = 163; \quad p_2 = 509; \quad p_3 = 603; \quad N = p_1^5 p_2^5 p_3^5$$

$$N = 340274423051874795558305386758572502851$$

(This has the same number of decimal digits as 2^{128} with the most significant four digits coinciding with it).

To get a lower bound for the number of keys one can compute the number of polynomials P_1, P_2, P_3, P_4 of degree at most 5, with P_i involving i variables, and the number of choices for 5-tuples of non-zero scalars. By Lemma 2, for degrees less than the finite field order, the automorphisms would be distinct, and the table in Lemma 3 says the lower bound is

$$(603 \times 509 \times 163)^{209} \times (602 \times 508 \times 162)^5.$$

This works out to be of the order of 2^{5478} or 10^{1649} . This is much larger than the keyspace of size 2^{128} from AES-128, despite a smaller block size!

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